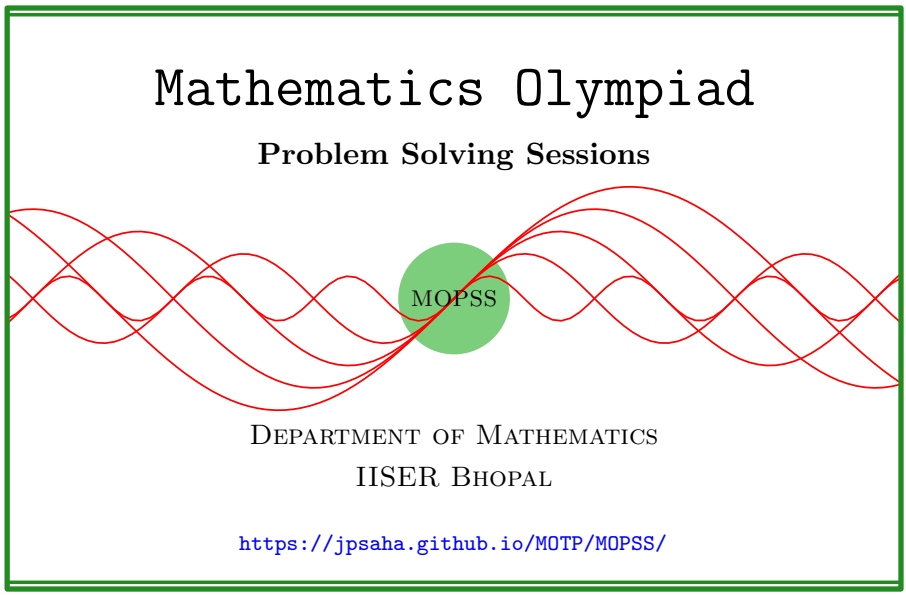
Induction

MOPSS



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Induction

Example 1.1 (India RMO 1993 P2)**.** Prove that the ten's digit of any power of 3 is even.

Solution 1. For any positive integer n, note that

$$\begin{aligned} 3^{2n} &= (10-1)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} 10^{n-i}(-1)^i \\ &\equiv \binom{n}{n-1} 10(-1)^{n-1} + (-1)^n \pmod{100} \\ &\equiv 10(-1)^{n-1}n + (-1)^n \pmod{100} \\ &\equiv \begin{cases} (10-n)10+1 \pmod{100} & \text{if } n \text{ is even,} \\ (n-1)10+9 \pmod{100} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Writing $n = 10k + r$ for some integers k, r with $1 \le r \le 10$, it follows that any power of 3 with a positive even exponent has an even number in its ten's digit.

For any positive integer n, observe that

$$3^{2n+1} \equiv \begin{cases} 3(10-n)10+3 \pmod{100} & \text{if } n \text{ is even,} \\ (3(n-1)+2)10+7 \pmod{100} & \text{if } n \text{ is odd.} \end{cases}$$

Also note that the ten's digit of 3 is even. It follows that any power of 3 with a positive odd exponent has an even number in its ten's digit. ■

Here is a proof of Euclid's theorem on the infinitude of primes by Saidak.

Example 1.2 (Infinitude of primes)**.** [Sai06] Let $a_1 = 2$ and $a_{n+1} = a_n(a_n + 1)$ for any positive integer n. Show that a_n has at least n distinct prime factors for any positive integer n.

Walkthrough —

(a) Show that $a_n \ge 2$ for any integer $n \ge 1$.

(b) Note that the integers $a_n, a_n + 1$ have no common prime factor.

(c) Conclude using induction.

Solution 2. ■

Example 1.3. Show that there are infinitely primes of the form $4n + 3$.

Exercise 1.4 (BStat-BMath 2012 P5, AoPS)**.** Let m be a natural number with digits consisting entirely of 6's and 0's. Prove that m is not the square of a natural number.

Walkthrough —

(a) Note that if any such number is a perfect square, then its last digit cannot be 6, that is, it is not congruent to 6 modulo 10, because no square is congruent to any of $6, 66$ modulo 100.

(b) It follows that if any such number is perfect square, then it is divisible by 100.

(c) Apply induction (on what?). A crucial step would be frame an inductive statement appropriately.

Exercise 1.5 (IOQM 2023 P10, AoPS)**.** The sequence $\langle a_n \rangle_{n \geq 0}$ is defined by $a_0 = 1, a_1 = -4$ and

$$a_{n+2} = -4a_{n+1} - 7a_n$$

for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

Summary — Compute a_n for small value of n, and then evaluate $a_n^2 - a_{n-1}a_{n+1}$ for small values of n. This may suggest a suitable pattern for $a_n^2 - a_{n-1}a_{n+1}$, which can be established by induction. Finally, compute $a_{50}^2 - a_{49}a_{51}$ and find its number of positive integer divisors.

Walkthrough —

(a) Compute the first few terms of the sequence to note that

$$\begin{aligned} a_0 &= 1, \\ a_1 &= -4, \\ a_2 &= 9, \\ a_3 &= -8, \\ a_4 &= -31, \end{aligned}$$

which gives

$$\begin{aligned} a_1^2 - a_0a_2 &= 16 - 9 \\ &= 7, \end{aligned}$$

$$\begin{aligned} a_2^2 - a_1a_3 &= 81 - 32 \\ &= 49, \\ a_3^2 - a_2a_4 &= 64 + 279 \\ &= 343. \end{aligned}$$

This may suggest that

$$a_n^2 - a_{n-1}a_{n+1} = 7^n$$

holds for any integer $n \geq 1$.

Solution 3. Let us establish the claim below.

Claim — For any integer $n \geq 1$,

$$a_n^2 - a_{n-1}a_{n+1} = 7^n.$$

Proof of the Claim. Note that it holds for $n = 1$. Assume that $n \geq 1$ is a positive integer such that

$$a_n^2 - a_{n-1}a_{n+1} = 7^n$$

holds. Observe that

$$\begin{aligned} a_{n+1}^2 - a_na_{n+2} &= a_{n+1}^2 - a_n(-4a_{n+1} - 7a_n) \\ &= a_{n+1}^2 + 4a_na_{n+1} + 7a_n^2 \\ &= a_{n+1}^2 + 4a_na_{n+1} + 7a_{n-1}a_{n+1} + 7^{n+1} \\ &= a_{n+1}(a_{n+1} + 4a_n + 7a_{n-1}) + 7^{n+1} \\ &= 7^{n+1}. \end{aligned}$$

By induction, it follows that This proves that

$$a_n^2 - a_{n-1}a_{n+1} = 7^n$$

holds for any integer $n \geq 1$. □

Consequently,

$$a_{50}^2 - a_{49}a_{51} = 7^{50},$$

whose number of positive integer divisors is equal to 51. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Sai06**] Filip Saidak. A new proof of Euclid's theorem. In: *Amer. Math. Monthly*, **113**:10 (2006), pp. 937–938. ISSN: 0002-9890. DOI: 10.2307/27642094. URL: http://dx.doi.org/10.2307/27642094 (cited p. 2)