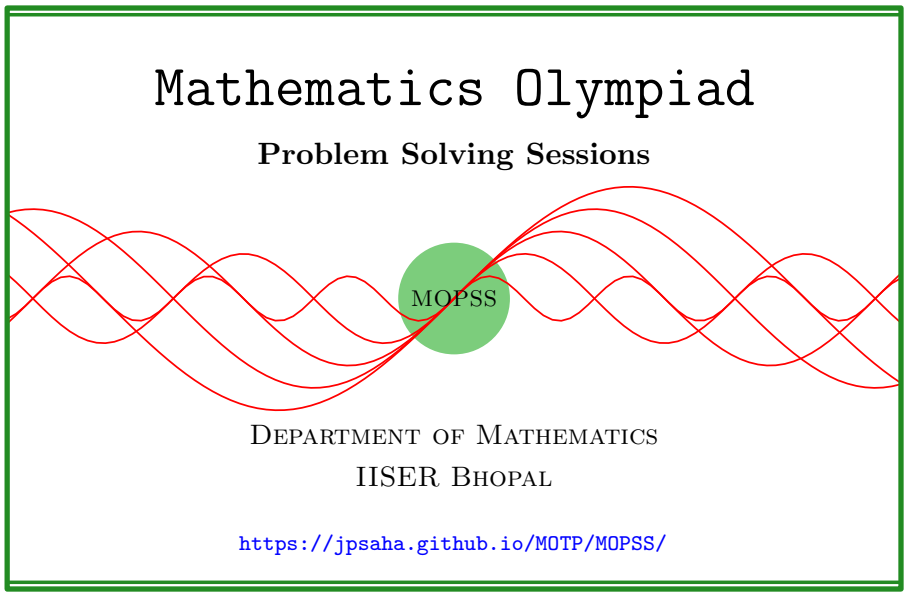
Counting in two different ways

MOPSS



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Counting in two different ways

See [**AF13**, §7], [**AE11**, §3.3].

Example 1.1. Find the sum of all distinct four digit numbers that can be formed using the digits $1, 2, 3, 4, 5$, each digit appearing at most once.

Solution 1. Let a, b, c, d be four distinct elements of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then among the four digit numbers that can be formed using a, b, c, d with no repeated digits, there are exactly six numbers with i in its j's place for any $i \in \{a, b, c, d\}, j \in \{1, 10, 100, 1000\}$. So the sum of such numbers is equal to $(a + b + c + d) \times (6 + 60 + 600 + 6000) = 6666(a + b + c + d)$. This shows that the required sum is equal to

$$\sum_{\{a,b,c,d\} \subseteq \{1,2,\ldots,9\}} 6666(a + b + c + d).$$

Note that

$$\begin{aligned}
\sum_{\substack{A \subseteq \{1,2,\ldots,9\} \\ |A|=4}} \sum_{a \in A} a &= \sum_{a=1}^{9} \sum_{\substack{A \subseteq \{1,2,\ldots,9\} \\ |A|=4, a \in A}} a \\
&= \sum_{a=1}^{9} \binom{8}{3} a \\
&= 45 \times \binom{8}{3} \\
&= 25200.
\end{aligned}$$

Hence, the required sum is equal to

$$6666 \times 25200.$$

■

Example 1.2 (India BStat-BMath 2014)**.** A class has 100 students. Let a_i, $1 \leq i \leq 100$, denote the number of friends the i-th student has in the class. For each $0 \leq j \leq 99$, let c_j denote the number of students having at least j friends. Show that

$$a_1 + a_2 + \cdots + a_{100} = c_1 + c_2 + \cdots + c_{99}.$$

Solution 2. For $1 \leq i \leq 100$, denote the i-th student by s_i. For $1 \leq j \leq 99$, let C_j denote the set of students having at least j friends. Note that for any $1 \leq i \leq 100$,

$$a_i = \sum_{j=1}^{99} 1_{C_j}(s_i)$$

holds, where for $1 \leq j \leq 99$, 1_{C_j} denotes the map, defined on $\{s_1, s_2, \ldots, s_{100}\}$, given by

$$1_{C_j}(s_i) = \begin{cases} 1 & \text{if } s_i \text{ lies in } C_j, \\ 0 & \text{otherwise.} \end{cases}$$

Summing over $1 \leq i \leq 100$, and interchanging the order of summation, we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{100} &= \sum_{j=1}^{99} \sum_{i=1}^{100} 1_{C_j}(s_i) \\ &= \sum_{j=1}^{99} |\{s_i \mid s_i \in C_j\}| \\ &= \sum_{j=1}^{99} c_j. \end{aligned}$$

This completes the proof. ■

> **Remark 1.** The following is **somewhat naive**, and **does require** additional explanation to be included (at which step(s)?). The following explains (with some effort from readers' end, of course!) why the stated result should hold, and it may also help to arrive at the above solution. However, the following lacks some details.

Walkthrough —

	c_1 bullets	c_2 bullets	c_3 bullets	$\cdots$	c_j bullets	$\cdots$	
	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$	$\cdots$	
s_1	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\cdots$	$\leftarrow a_1$ bullets
s_2	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\cdots$	$\leftarrow a_2$ bullets
s_3	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\cdots$	$\leftarrow a_3$ bullets
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
s_i	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\cdots$	$\leftarrow a_i$ bullets
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
s_{100}	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\cdots$	$\leftarrow a_{100}$ bullets

Denote the students by $s_1, s_2, \ldots, s_{100}$ and write $s_1, s_2, \ldots, s_{100}$ in a vertical column. Then for each $1 \le i \le 100$, put a_i bullets next to s_i (as shown above). Then the sum $a_1 + \cdots + a_{100}$ is equal to the total number of bullets. It turns out that the number of bullets in the j-th column is equal to c_j for any $1 \le j \le 99$, proving that the total number of bullets is also equal to $c_1 + c_2 + c_3 + \cdots + c_{99}$.

Example 1.3 (India RMO 2019a P6)**.** Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that $\gcd(a, b) = \gcd(b, c) = \gcd(c, d) = \gcd(d, a) = 1$.

Solution 3. Denote the chosen 91 integers by $n_1, \ldots, n_{91}$. For $1 \le i \le 91$, let d_i denote the number of integers among them which are relatively prime with n_i. Consider the triples (a, b, c) where a, b, c run over the chosen 91 integers, and satisfy $\gcd(a, b) = \gcd(a, c) = 1$. Counting these triples along their first entry, it follows that the number of such triples is

$$\binom{d_1}{2} + \cdots + \binom{d_{91}}{2}.$$

If the statement to be proved is false, then the given any two distinct integers b, c among them, it can be completed to at most one such triple (a, b, c). This shows that

$$\binom{d_1}{2} + \cdots + \binom{d_{91}}{2} \le \binom{91}{2},$$

which implies that

$$\begin{aligned}
\binom{91}{2} &\ge \frac{1}{2}\left(d_1^2 + \cdots + d_{91}^2 - d_1 - \cdots - d_{91}\right) \\
&= \frac{1}{2}\left(d_1^2 + \cdots + d_{91}^2 - 2 \cdot 456\right) \\
&\ge \frac{1}{2}\left(\frac{(d_1 + \cdots + d_{91})^2}{91} - 912\right) \quad \text{(using the Cauchy–Schwartz inequality)}
\end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

$$= \frac{1}{2}\left(\frac{(2\cdot 456)^2}{91} - 912\right)$$
$$= 456 \cdot 9,$$

and this yields $910 > 912$, which is impossible. This completes the proof. ■

Example 1.4 (India TST 2001?)**.** [Sri14, Chapter 6, Example 9] Let G be a graph on n vertices, having e edges, and no 4-cycles. Show that

$$e \leq \frac{n}{4}(1 + \sqrt{4n-3}).$$

Summary — A similar counting as above yields

$$\binom{n}{2} \geq \frac{1}{2}\left(\frac{(2e)^2}{n} - 2e\right),$$

which implies the above bound.

Exercise 1.5 (IMOSL 1995 N5, AoPS)**.** At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

Walkthrough —

(a) Count the number of triples (a, b, c) where a exchanges greetings with b and c in two different ways.

(b) Show that $12k - 1$ divides 525 and deduce the value of k.

Solution 4. Let λ denote the integer such that for any two people, precisely λ persons exchange greetings with them. Counting the number of triples of the form (a, b, c) where a exchanges greetings with b and c, we obtain

$$12k\binom{3k+6}{2} = \lambda\binom{12k}{2},$$

which yields

$$\lambda = \frac{(3k+5)(3k+6)}{12k-1}$$
$$= \frac{1}{4}\frac{36k^2+132k+120}{12k-1}$$
$$= \frac{1}{4}\left(3k+11+\frac{3k+131}{12k-1}\right)$$

$$= \frac{3k+11}{4} + \frac{1}{16}\left(1 + \frac{525}{12k-1}\right).$$

This shows that $12k-1$ divides 525 and the quotient $525/(12k-1)$ is congruent to 3 modulo 4, that is, for some integer $\ell \geq 0$, we have

$$525 = (12k-1)(4\ell+3).$$

Noting that $525 = 3 \cdot 5^2 \cdot 7$, it follows that one of $12k-1, 4\ell+3$ is a multiple of 3 and the other one is a multiple of 7. This shows that $12k-1$ is equal to one of

$$3, 15, 45, 7, 35, 105,$$

and hence $12k-1 = 35$, which gives $k = 3$. It follows that there are 36 people at the meeting. ■

Example 1.6 (India RMO 2023a P4)**.** The set X of N four-digit numbers formed from the digits $1, 2, 3, 4, 5, 6, 7, 8$ satisfies the following condition: *for any two different digits from* $1, 2, 3, 4, 5, 6, 7, 8$*, there exists a number in* X *which contains both of them.* Determine the smallest possible value of N.

First, let's work on it. Let X be a set satisfying the required conditions and it has the minimum cardinality among such sets. Given an element x of X, we may assume that x has distinct digits. Otherwise, any repetition of a digit can be replaced by an integer from $\{1, \ldots, 8\}$ which does not occur as a digit in x. Note that after this modification of an element of X, the modified set has the same cardinality as the initial set, and it continues to have the property that for any two different digits from $\{1, \ldots, 8\}$, there exists a number in the modified set which contains both of them. Thus, modifying the elements of X if required, we may assume that the digits of the elements of X are distinct. By the hypothesis on the cardinality of X, it follows that no two elements of X are equal up to permutation. So, the elements of X can be considered as subsets of $\{1, 2, \ldots, 8\}$ of size 4, and X can be thought as a set of certain size 4 subsets of $\{1, \ldots, 8\}$ such that any size two subsets of $\{1, \ldots, 8\}$ is contained in some element of X.

Consider the set

$$A := \{(P, Q) \mid |P| = 2, Q \in X, P \subseteq Q\}.$$

Note that the size of A is $\binom{4}{2}|X| = 6|X|$. By hypothesis, it follows that A contains at least $\binom{8}{2} = 28$ elements. Hence, X contains at least 5 elements.

With some effort, one can find the following set

$$\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{1, 5, 2, 6\}, \{3, 7, 4, 8\}, \{1, 7, 8, 2\}, \{3, 5, 6, 4\}\}.$$

So far, we have proved that X contains at least 5 elements. Based on this information and the above example only (possibly together with the fact that

we are not able to come up with a set consisting of five size 4 subsets of $\{1, \ldots, 8\}$ having the stated property), we are not in a position to conclude that X has cardinality 6.

In fact, a modification of the above argument does prove that $|X| \geq 6$. Indeed, consider the set

$$A' := \{(P, Q) \,|\, |P| = 1, Q \in X, P \subseteq Q\}.$$

Note that the size of A' is $4|X|$. Also note that for any element $i \in \{1, \ldots, 8\}$, it is contained in at least three elements of X. Indeed, there are 7 size two subsets of $\{1, \ldots, 8\}$ containing i, and the union of no two size 4 subsets of $\{1, \ldots, 8\}$ contains all the size two subsets of $\{1, \ldots, 8\}$ containing i. By hypothesis, it follows that A' contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements.

Note that the size of A' is $4|X|$. Also note that for any element $i \in \{1, \ldots, 8\}$, it is contained in at least three elements of X. Indeed, there are 7 size two subsets of $\{1, \ldots, 8\}$ containing i, and the union of no two size 4 subsets of $\{1, \ldots, 8\}$ contains all the size two subsets of $\{1, \ldots, 8\}$ containing i. By hypothesis, it follows that A' contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements. ♣

Solution 5. Let X be a set satisfying the required conditions and it has the minimum cardinality among such sets. Given an element x of X, we may assume that x has distinct digits. Otherwise, any repetition of a digit can be replaced by an integer from $\{1, \ldots, 8\}$ which does not occur as a digit in x. Note that after this modification of an element of X, the modified set has the same cardinality as the initial set, and it continues to have the property that for any two different digits from $\{1, \ldots, 8\}$, there exists a number in the modified set which contains both of them. Thus, modifying the elements of X if required, we may assume that the digits of the elements of X are distinct. By the hypothesis on the cardinality of X, it follows that no two elements of X are equal up to permutation. So, the elements of X can be considered as subsets of $\{1, 2, \ldots, 8\}$ of size 4, and X can be thought as a set of certain size 4 subsets of $\{1, \ldots, 8\}$ such that any size two subsets of $\{1, \ldots, 8\}$ is contained in some element of X.

Consider the set

$$A := \{(P, Q) \,|\, |P| = 1, Q \in X, P \subseteq Q\}.$$

Note that the size of A is $4|X|$. Also note that for any element $i \in \{1, \ldots, 8\}$, it is contained in at least three elements of X. Indeed, there are 7 size two subsets of $\{1, \ldots, 8\}$ containing i, and the union of no two size 4 subsets of $\{1, \ldots, 8\}$ contains all the size two subsets of $\{1, \ldots, 8\}$ containing i. By hypothesis, it follows that A contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements.

Note that the set

$$\{\{1,2,3,4\},\{5,6,7,8\},\{1,5,2,6\},\{3,7,4,8\},\{1,7,8,2\},\{3,5,6,4\}\}$$

has size 6 and it consists of certain size 4 subsets of $\{1,\ldots,8\}$ such that any size two subset of $\{1,\ldots,8\}$ is contained in one such size 4 subset.

This proves that the smallest possible value of N is 6. ■

Solution 6. Note that the following set of four-digit numbers satisfy the required condition.

$$\{1234, 5678, 1256, 3478, 1278, 3456\}$$

This shows that the smallest possible value of N is at most 6.

Let X be a set of N four-digit numbers satisfying the given condition. Note that for any integer $1 \leq i \leq 8$, two four-digit numbers containing i as a digit, together contains at most 7 distinct elements of $\{1,2,\ldots,8\}$ as digits. Hence, for any $1 \leq i \leq 8$, there are at least three elements in X which contain i as a digit. Counting the number of pairs of the form (i,x), where $1 \leq i \leq 8$ and x is an element of X containing i as a digit, we obtain

$$4N \geq 8 \cdot 3,$$

which yields $N \geq 6$.

This proves that the smallest possible value of N is 6. ■

> **Remark.** Is it helpful to count the pairs of the form $(\{i,j\},x)$ where $1 \leq i < j \leq 8$ and x is an element of X containing i,j as digits?

References

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[**AF13**] T. Andreescu and Z. Feng. *A Path to Combinatorics for Undergraduates: Counting Strategies*. Birkhäuser Boston, 2013. ISBN: 9780817681548. URL: https://books.google.de/books?id=3mwQBwAAQBAJ (cited p. 2)

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Sri14**] Pranav Sriram. "Olympiad Combinatorics". Available at https://www.aops.com/community/c6h601134. 2014 (cited p. 5)