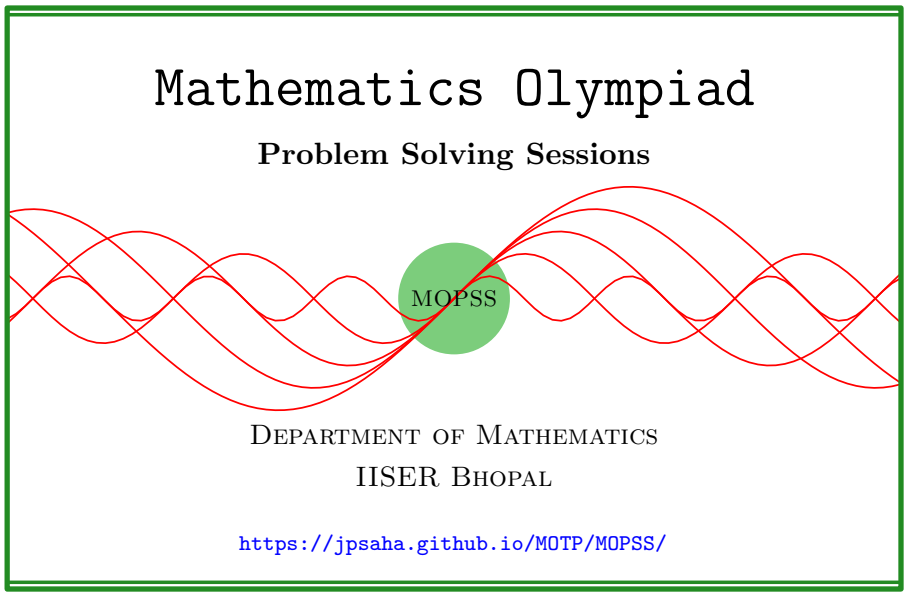
Viète's relations

MOPSS



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Viète's relations

Example 1.1 (India RMO 2012e P2)**.** cf. [GA17, Problem 141] Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of P are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

Solution 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of P. Note that

$$\begin{aligned}
&\sum_{1\leq i,j\leq n, i\neq j} (\alpha_i - \alpha_j)^2 \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 2\sum_{1\leq i,j\leq n, i\neq j} \alpha_i\alpha_j \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2(n-1)\left(\sum_{1\leq i\leq n} \alpha_i\right)^2 - 4(n-1)\left(\sum_{1\leq i<j\leq n} \alpha_i\alpha_j\right) - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\binom{n}{2} \\
&= 0.
\end{aligned}$$

Since $\alpha_1, \ldots, \alpha_n$ are real, it follows that they are all equal. Using $\alpha_1 + \cdots + \alpha_n = n$, we get

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1.$$

This implies that $a_i = (-1)^i\binom{n}{i}$ for any $0 \leq i \leq n-1$. ■

Solution 2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of $P(x)$. Note that

$$\begin{aligned}
&(\alpha_1 - 1)^2 + (\alpha_2 - 1)^2 + \cdots + (\alpha_n - 1)^2 \\
&= \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \\
&= (\alpha_1 + \alpha_2 + \cdots + \alpha_n)^2 - 2\sum_{1\leq i<j\leq n} \alpha_i\alpha_j + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n)
\end{aligned}$$

$$= n^2 - n(n-1) + n - 2n$$
$$= 0.$$

So the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all equal to 1. This implies that $P(x) = (x-1)^n$, and hence a_i is equal to $(-1)^i \binom{n}{i}$ for any $0 \leq i \leq n$. ■

Exercise 1.2 (All-Russian Mathematical Olympiad 2007 Grade 11 Day 2 P6, AoPS, by N. Agakhanov, I. Bogdanov)**.** Do there exist nonzero reals a, b, c such that, for any $n > 3$, there exists a polynomial $P_n(x) = x^n + \cdots + ax^2 + bx + c$, which has exactly n (not necessarily distinct) integral roots?

Walkthrough —

(a)

Solution 3. Assume that there exist nonzero reals a, b, c satisfying the given condition. Let $n > 3$ be a positive integer, and let $P_n(x) = x^n + \cdots + ax^2 + bx + c$ be a polynomial with exactly n integral roots, which we denote by $r_1, r_2, \ldots, r_n$, counted with multiplicities. Since c is nonzero, none of the roots $r_1, r_2, \ldots, r_n$ is equal to zero. It follows that

$$\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n}$$

are the roots of the polynomial

$$Q_n(x) = x^n P\left(\frac{1}{x}\right) = cx^n + bx^{n-1} + ax^{n-2} + \cdots + x + 1.$$

By Vieta's formulas, we obtain

$$\frac{1}{r_1} + \cdots + \frac{1}{r_n} = -\frac{b}{c}, \quad \sum_{1 \leq i < j \leq n} \frac{1}{r_i r_j} = \frac{a}{c}.$$

This implies that

$$\left(\frac{1}{r_1} + \cdots + \frac{1}{r_n}\right)^2 - 2 \sum_{1 \leq i < j \leq n} \frac{1}{r_i r_j} = \frac{b^2 - 2ac}{c^2},$$

and hence

$$\frac{1}{r_1^2} + \cdots + \frac{1}{r_n^2} = \frac{b^2 - 2ac}{c^2}.$$

Also note that

$$r_1 r_2 \cdots r_n = (-1)^n c.$$

It follows that

$$b^2 - 2ac \geq \frac{r_1^2 r_2^2 \cdots r_n^2}{r_1^2} + \cdots + \frac{r_1^2 r_2^2 \ldots r_n^2}{r_n^2} \geq n.$$

This shows that $n \leq b^2 - 2ac$ for any integer $n > 3$, which is a contradiction.

Therefore, there do not exist nonzero reals a, b, c satisfying the given condition. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**GA17**] Răzvan Gelca and Titu Andreescu. *Putnam and beyond*. Second. Springer, Cham, 2017, pp. xviii+850. ISBN: 978-3-319-58986-2; 978-3-319-58988-6. DOI: 10.1007/978-3-319-58988-6. URL: https://doi.org/10.1007/978-3-319-58988-6 (cited p. 2)