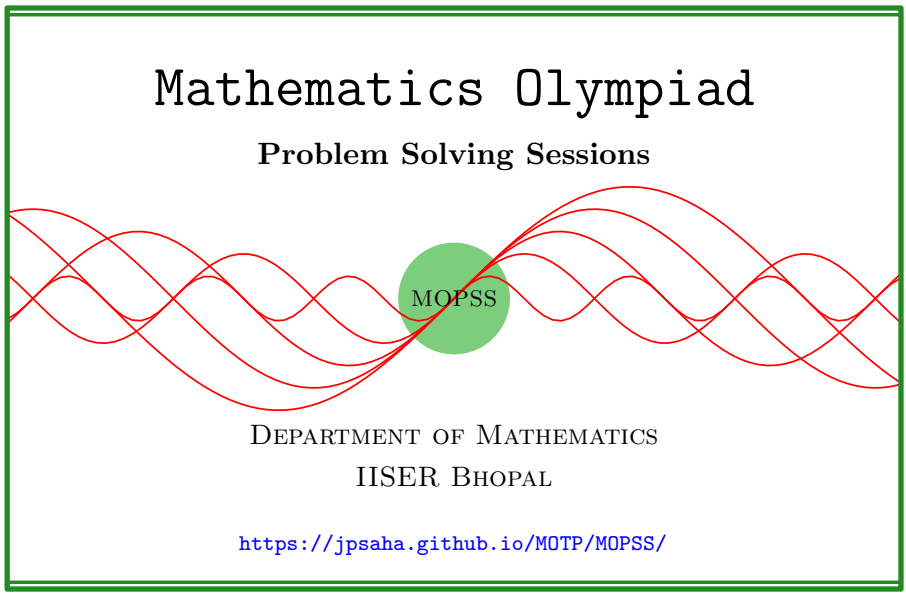
Crossing the x-axis

MOPSS



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Crossing the x-axis

Here are a few problems from this notes, and this one.

Example 1.1. Suppose $P(x)$ is a polynomial with real coefficients such that $P(x) = x$ has no real solution. Show that $P(P(x)) = x$ has no real solutions.

> **Walkthrough** —
>
> **(a)**

Solution 1. Since $x \mapsto P(x)$ defines a continuous map from $\mathbb{R} \to \mathbb{R}$, by the intermediate value theorem, it follows that $P(x) > x$ holds for all $x \in \mathbb{R}$ or $P(x) < x$ holds for all $x \in \mathbb{R}$. If $P(x) > x$ holds for all $x \in \mathbb{R}$, then $P(P(x)) > P(x) > x$ holds for all $x \in \mathbb{R}$, and hence $P(P(x)) = x$ has no real solutions. Similarly, if $P(x) < x$ holds for all $x \in \mathbb{R}$, then $P(P(x)) = x$ has no real solutions. ■

Exercise 1.2 (All-Russian Mathematical Olympiad 2013 Grade 9 Day 1 P1, AoPS, by I. Bogdanov)**.** Given three distinct real numbers a, b, and c, show that at least two of the three following equations

$$(x-a)(x-b) = x-c,$$
$$(x-b)(x-c) = x-a,$$
$$(x-c)(x-a) = x-b$$

have real solutions.

Walkthrough —
(a)

Solution 2. Without loss of generality, let us consider the case that the equations

$$(x-a)(x-b) = x-c, (x-b)(x-c) = x-a$$

do not any real root. It follows that their discriminants are negative, that is,

$$(a+b+1)^2 - 4(ab+c), (b+c+1)^2 - 4(bc+a)$$

are negative. Note that

$$\begin{aligned}(a+b+1)^2 - 4(ab+c) &= (a-b)^2 + 1 + 2(a+b-2c)\\ &= (b-a+1)^2 + 4(a-c),\\ (b+c+1)^2 - 4(bc+a) &= (b-c)^2 + 1 + 2(b+c-2a)\\ &= (b-c+1)^2 + 4(c-a).\end{aligned}$$

Since a and c are distinct real numbers, one of $a-c$ and $c-a$ is positive. Thus, the discriminant of one of the equations

$$(x-a)(x-b) = x-c, (x-b)(x-c) = x-a$$

is non-negative, which is a contradiction. Therefore, at least two of the three equations have real solutions. ■

Solution 3. Let $f(x)$, $g(x)$, and $h(x)$ denote the polynomials defined by

$$\begin{aligned}f(x) &= (x-a)(x-b) - (x-c),\\ g(x) &= (x-b)(x-c) - (x-a),\\ h(x) &= (x-c)(x-a) - (x-b).\end{aligned}$$

Note that

$$\begin{aligned}f(x) + g(x) &= (x-b)(2x-a-c) - (2x-a-c)\\ &= (2x-a-c)(x-b-1).\end{aligned}$$

This implies that $f(x) + g(x)$ admits a real root. Hence, at least one of $f(x)$ and $g(x)$ is non-positive at some real number. If one of them admits a real root, then we are done. Otherwise, at least one of them is negative at some real number. Also note that if k is a real number satisfying $k \geq |a| + |b| + |c| + 2$, then

$$\begin{aligned}f(k + |a| + |b|) &= (k + |a| + |b| + a)(k + |a| + |b| + b) - (k + |a| + |b| + c)\\ &\geq k^2 - k - |a| - |b| - |c|\end{aligned}$$

$$\begin{aligned} &\geq k - |a| - |b| - |c| \\ &> 0 \end{aligned}$$

holds, and similarly,

$$g(k + |a| + |b|) > 0$$

holds too. By the intermediate value theorem, at least one of $f(x)$ and $g(x)$ admits a real root. Interchanging the roles of f, g, and h, it follows that at least one of any two of the polynomials f, g, and h admits a real root. Therefore, at least two of the three equations have real solutions for x. ■

Exercise 1.3 (All-Russian Mathematical Olympiad 2007 Grade 8 Day 1 P1, AoPS, by O. Podlipsky)**.** Given reals numbers a, b, c, prove that at least one of three equations

$$\begin{aligned} x^2 + (a-b)x + (b-c) &= 0, \\ x^2 + (b-c)x + (c-a) &= 0, \\ x^2 + (c-a)x + (a-b) &= 0 \end{aligned}$$

has a real root.

Walkthrough —

(a)

Solution 4. Note that the sum of the discriminants of the quadratic polynomials

$$x^2 + (a-b)x + (b-c), x^2 + (b-c)x + (c-a), x^2 + (c-a)x + (a-b)$$

are equal to

$$\begin{aligned} &(a-b)^2 - 4(b-c) + (b-c)^2 - 4(c-a) + (c-a)^2 - 4(a-b) \\ &= (a-b)^2 + (b-c)^2 + (c-a)^2, \end{aligned}$$

which is non-negative, and hence, at least one of the discriminants is non-negative. This implies that at least one of the three equations has a real root. ■

Solution 5. Consider the three quadratic polynomials defined by

$$\begin{aligned} f(x) &= x^2 + (a-b)x + (b-c), \\ g(x) &= x^2 + (b-c)x + (c-a), \\ h(x) &= x^2 + (c-a)x + (a-b). \end{aligned}$$

Note that

$$f(0) + g(0) + h(0) = 0,$$

which implies that at least one of $f(0)$, $g(0)$, $h(0)$ is non-positive. Without loss of generality, assume that $f(0) \leq 0$. If $f(0)$ is zero, then $x = 0$ is a root of $f(x)$. If $f(0) < 0$, then noting that

$$\begin{aligned}
&f(2|a-b| + 2|b-c| + 1) \\
&= \frac{1}{2}(2|a-b| + 2|b-c| + 1)^2 + (a-b)(2|a-b| + 2|b-c| + 1) \\
&\quad + \frac{1}{2}(2|a-b| + 2|b-c| + 1)^2 + (b-c) \\
&= \frac{1}{2}(2|a-b| + 2|b-c| + 1)\,(2|a-b| + 2|b-c| + 1 - 2(a-b)) \\
&\quad + \frac{1}{2}(2|a-b| + 2|b-c| + 1)^2 + (b-c) \\
&\geq 1
\end{aligned}$$

holds, it follows from the intermediate value theorem that $f(x)$ has a real root. ■

Exercise 1.4 (All-Russian Mathematical Olympiad 2007 Grade 9 Day 1 P1, AoPS, by S. Berlov)**.** Let $f(x)$, $g(x)$ be monic quadratic polynomials with real coefficients such that the equations $f(g(x)) = 0$ and $g(f(x)) = 0$ have no real roots. Prove that at least one of the equations $f(f(x)) = 0$ and $g(g(x)) = 0$ has no real roots.

Walkthrough —

(a)

Solution 6. If one of the polynomials $f(x)$, $g(x)$ has no real roots, then the conclusion follows immediately. Thus, we may assume that both $f(x)$ and $g(x)$ have real roots. Denote the roots of $f(x)$ by α_1, α_2, and the roots of $g(x)$ by β_1, β_2. We need to show that none of α_1, α_2 lies in $f(\mathbb{R})$, or none of β_1, β_2 lies in $g(\mathbb{R})$.

Claim — If $\alpha_2 \leq \beta_2$, then none of α_1, α_2 lies in $f(\mathbb{R})$.

Proof of the Claim. Let us assume that $\alpha_1 \leq \beta_2$. If some of α_1, α_2 lies in $f(\mathbb{R})$, then noting that f is a monic quadratic polynomial, it follows from the intermediate value theorem that β_2 lies in $f(\mathbb{R})$, which shows that the equation $g(f(x)) = 0$ has a real root, a contradiction. This proves the claim. □

If $\alpha_2 \leq \beta_2$, then by the Claim, the equation $f(f(x)) = 0$ has no real roots. Similarly, if $\beta_2 \leq \alpha_2$, then the equation $g(g(x)) = 0$ has no real roots. This completes the proof. ■

Example 1.5. Any polynomial of odd degree with real coefficients has a real root.

Example 1.6. Let $P(x)$ and $Q(x)$ be monic polynomials of degree 10 having real coefficients. Assume that the equation $P(x) = Q(x)$ has no real roots. Prove that the equation $P(x+1) = Q(x-1)$ has at least one real root.

Walkthrough —

(a) Consider the difference $P(x) - Q(x)$ to show that the coefficient of x^9 in these polynomials are equal.

(b) Prove that the polynomial $P(x+1) - Q(x-1)$ is of degree 9.

Solution 7. Note that $P(x) - Q(x)$ is a polynomial of degree at most 9 having real coefficients. Since $P(x) - Q(x)$ has no real root, it follows that it has degree at most 8. In other words, the coefficients of x^9 in $P(x), Q(x)$ are the same. Note that $P(x+1) - Q(x-1)$ is of degree ≤ 9, and the coefficient of x^9 in $P(x+1) - Q(x-1)$ is equal to the coefficient of x^9 in $(x+1)^{10} - (x-1)^{10}$, which is equal to 20. This shows that $P(x+1) - Q(x-1)$ is a polynomial of degree 9 with real coefficients. Consequently, it has at least one real root. ■

Example 1.7. Let $P(x)$ be a nonconstant polynomial with real coefficients having a real root. Suppose it does not vanish at 0. Show that the monomial terms appearing in $P(x)$ can be erased one by one to obtain its constant term such that the intermediate polynomial have at least one real root.

Walkthrough —

(a)

Solution 8. Write $P(x) = a_n x^n + \cdots + a_0$ with $a_n \ldots, a_0$ lying in $\mathbb{R}$, and $a_n a_0 \neq 0$. Since any polynomial of odd degree has a real root, it follows that if the degree of $P(x)$ is odd, then the nonconstant monomials, other than the leading term, can be erased one by one, and then, the leading term can be erased to obtain a_0, and the intermediate polynomials have a real root. If a_n, a_0 are of opposite signs and n is even, then the same process can be followed, and note that any of the intermediate polynomials takes values of opposite signs at 0 and at a large enough integer, and hence has a real root.

Let us assume that n is even, and a_n, a_0 are of the same sign. Let α denote a real root of $P(x)$. Let $Q(x)$ denote the polynomial $P(x) - a_n x^n$. Note that

$Q(0) = a_0$ and $Q(\alpha) = -a_n\alpha^n$. Since a_0 is nonzero, it follows that $\alpha \neq 0$, and hence, $Q(0), Q(\alpha)$ are of opposite signs. This shows that $Q(x)$ is a nonconstant polynomial with real coefficients having a real root, and it does not vanish at 0. Since the degree of $Q(x)$ is smaller than that of $P(x)$, by induction, we are done. ■

Exercise 1.8 (China TST 1995 Day 2 P2, AoPS)**.** Alice and Bob play a game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \cdots + \square x + 1.$$

They take turns to fill the empty boxes. If the resulting polynomial has no real root, Alice wins, otherwise, Bob wins. If Alice goes first, who has a winning strategy?

Walkthrough —

(a) There are more odds than evens among the integers $1, 2, \ldots, 2n-1$.

(b) Can Bob have a winning strategy using the odds in his favour?

Solution 9. Bob has a winning strategy, as described below.

Bob makes sure that at the end of each of his turns except the last one, the number of even powers of x whose coefficients have been provided by some of them is equal to the number of odd powers of x whose coefficients have been provided by some of them. This can be done, for instance, if during a turn of Bob, other than the last turn, Bob provides the coefficient of an odd (resp. even) power of x if Alice has provided the coefficient of an even (resp. odd) power of x in the preceeding turn.

Since $n \geq 2$, it follows that Bob gets at least one turn. At the beginning of the final turn of Bob, there are two powers of x whose coefficients are to be determined, denote them by x^i, x^j, their coefficients by c_i, c_j respectively. Let $Q(x)$ denote the polynomial, obtained by the erasing the terms corresponding to x^i, x^j from the polynomial that Bob had at the beginning of his final turn. Note that

$$P(x) = Q(x) + c_i x^i + c_j x^j.$$

Note that at least of i, j is odd. Interchanging i, j if necessary, let us assume that i is odd. We descrie the strategy that Bob follows in the two cases below.

Let us consider the case that j is even. Bob determines c_j in such a way that for any choice of c_i, the completed polynomial $P(x)$ is guaranteed to have at least one real root. This can be done, for instance, by taking c_j satisfying

$$Q(1) + Q(-1) + 2c_j = 0.$$

For any choice of c_i, the above choice of c_j shows that $P(1) + P(-1) = 0$, which implies that $P(x)$ has a root in the interval $[-1, 1]$.

Let us consider the case that j is odd. Bob determines c_j in such a way that for any choice of c_i by Alice in the next turn, the completed polynomial $P(x)$ is guaranteed to have at least one real root. This can be done, for instance, by taking c_j satisfying

$$Q(2) + c_j 2^j + 2^i Q(-1) - c_j 2^i = 0.$$

Since $i \neq j$, the above holds for some $c_j \in \mathbb{R}$. For any choice of c_i, note that

$$P(2) + 2^i P(-1) = 0$$

holds, which implies that $P(x)$ has a root in $[-1, 2]$. ■

Exercise 1.9 (Bay Area MO 2004 P5, AoPS)**.** Find (with proof) all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions.

1. $f(0) = 2004$.
2. If x is irrational, then $f(x)$ is also irrational.

Solution 10. Let $f(x)$ be a polynomial satisfying the required conditions. Note that $f(x)$ is nonconstant. If $f(x)$ is linear, then it is equal to $x + 2004$. It remains to consider the case when the degree of $f(x)$ is at least 2, which we assume from now on.

Let n be a positive integer and p be a prime number satisfying

$$f(0) - n = -p.$$

Since $f(x) - n$ takes positive values for large positive x and a negative value at $x = 0$, by the intermediate value theorem, the polynomial $f(x) - n$ has a real root α. By the given conditions, α is rational. Since $f(x) - n$ is a monic polynomial with integer coefficients and its constant term is $-p$, by the rational root theorem, α is an integer and it divides p. Since p is prime, it follows that α is either 1 or -1 or p or $-p$. Write

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + f(0).$$

Note that for any integer k,

$$\begin{aligned} |f(k) - n| &\geq |f(0) - n| - |k^n + a_{n-1}k^{n-1} + \cdots + a_1 k| \\ &\geq p - \left(k^n + |a_{n-1}|k^{n-1} + \cdots + |a_1|k\right) \\ &\geq p - \left(1 + |a_{n-1}| + \cdots + |a_1|\right) k^n, \end{aligned}$$

and

$$|f(k) - n| \geq |k^n + a_{n-1}k^{n-1} + \cdots + a_1 k| - |f(0) - n|$$

The content posted here and at this blog by Evan Chen are quite useful.

$$\geq |k^n| - \left(|a_{n-1}||k^{n-1}| + \cdots + |a_1||k|\right) - p$$
$$\geq |k^n| - \left(|a_{n-1}| + \cdots + |a_1|\right)|k^{n-1}| - p$$

hold.

If p is large enough, then so is n, and hence $f(x) - n$ does not vanish at any of $1, -1, p, -p$. Indeed, if

$$p > 1 + |a_{n-1}| + \cdots + |a_1|,$$

then $f(1) - n$ and $f(-1) - n$ are nonzero, and $f(p) - n$ and $f(-p) - n$ have absolute values at least

$$p^{n-1}(p - (|a_{n-1}| + \cdots + |a_1|)) - p > p^{n-1} - p \geq 0,$$

where the last inequality holds since $n \geq 2$. This contradicts the fact that α is one of $1, -1, p, -p$. Therefore, no polynomial $f(x)$ of degree at least 2 satisfies the required conditions. This proves that any polynomial $f(x)$ satisfying the required conditions is linear, and is equal to $x + 2004$.

Also note that the polynomial $x+2004$ indeed satisfies the required conditions.

Hence, the only polynomial $f(x)$ satisfying the required conditions is

$$f(x) = x + 2004.$$

■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)