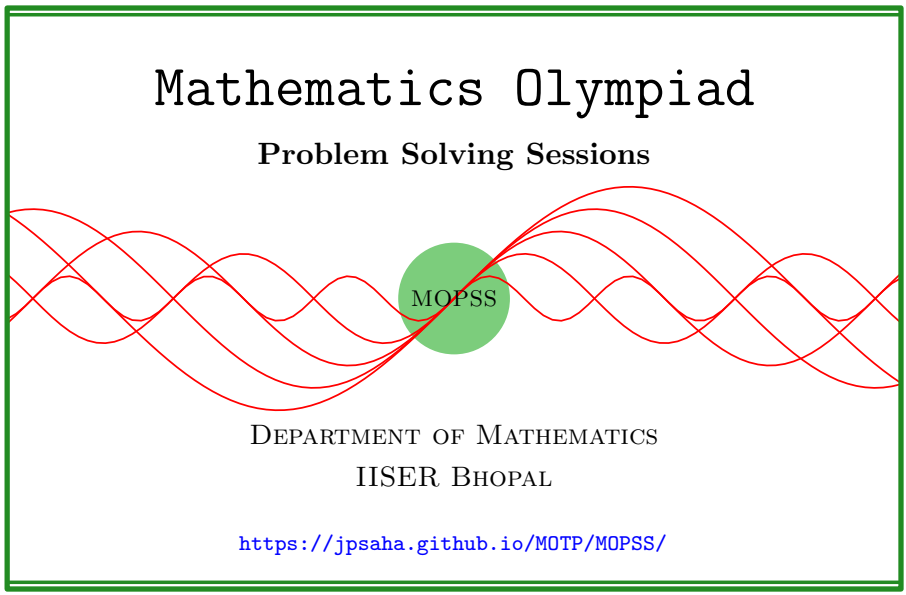
Functional equations

MOPSS



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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§1 Functional equations

Example 1.1 (cf. Canadian Mathematical Olympiad 1969 P8, Australian Mathematics Competition 1984, India Pre-RMO 2012 P16)**.** [**Tao06**, Problem 3.2] Suppose

$$x_1, x_2, x_3, x_4, \dots$$

is a sequence of integers satisfying the following properties:

(1) $x_2 = 2$,

(2) $x_{mn} = x_m x_n$ for all positive integers m, n,

(3) $x_m < x_n$ for any positive integers m, n with $m < n$.

Find x_{2024}. What is the value of the sum

$$x_1 + x_2 + \cdots + x_{2024}?$$

Summary — Observe that $x_{2^n} = 2^n$ for any $n \geq 1$. Combining this with the hypothesis that $\{x_n\}_{n\geq 1}$ is an increasing sequence of **positive integers**, conclude that $x_n = n$ for any $n \geq 1$.

Walkthrough —

(a) What can be said about x_4, x_8, x_{16}, x_{32}?

(b) Note that $x_4 = x_{2\times 2}, x_8 = x_{4\times 2}, x_{16} = x_{8\times 2}, x_{32} = x_{16\times 2}$.

(c) Can one show that $x_{2^n} = 2^n$ for any $n \geq 1$?

(d) Show that $x_m = m$ for any $m \geq 1$ (does property (3) help?).

Solution 1. From the second condition, we obtain

$$x_{2^n} = x_2^n$$

for any integer $n \geq 1$. Using the first condition, it follows by induction that

$$x_{2^n} = 2^n$$

for any integer $n \geq 1$. Since $\{x_n\}_{n\geq 1}$ is an increasing sequence of positive integers, it follows that $x_n = n$ for any positive integer n. This gives

$$x_{2024} = 2024.$$

It follows that

$$x_1 + x_2 + \cdots + x_{2024} = 1012 \cdot 2025 = 2049300.$$

■

Exercise 1.2 (Putnam 1963 A2, AoPS)**.** Let $f\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for every relatively prime pair of positive integers m and n. Show that $f(n) = n$ for every positive integer n.

Walkthrough —

(a)

Solution 2. Note that $f(1) = 1$ since $f(2) = f(1)f(2)$. Write $f(3) = k$. Note that

$$\begin{aligned} k(k+2) &\leq f(3)f(5) \\ &= f(15) \\ &\leq f(18) - 3 \\ &= f(2)f(9) - 3 \\ &= 2(f(10) - 1) - 3 \\ &= 2f(2)f(5) - 5 \\ &= 4f(5) - 5 \\ &\leq 4(f(6) - 1) - 5 \\ &= 4f(2)f(3) - 9 \\ &= 8k - 9 \end{aligned}$$

holds, which implies that $k^2 - 6k + 9 \leq 0$, which gives $k = 3$.

Claim — For any integer $n \geq 3$, we have $f(n) = n$.

Proof of the Claim. Note that $f(n) = n$ holds for $n = 3$. Assume that $m \geq 3$ is an integer such that $f(n) = n$ holds for all $1 \leq n \leq m$.

Let us first consider the case that m is odd. Then,

$$f(2m) = 2f(m) = 2m$$

holds. Since f is strictly increasing, using $f(m) = m$ and $f(2m) = 2m$, it follows that

$$f(m+1) = m+1.$$

Next, consider the case that m is even. Then, it follows that $m-1 \geq 3$ and $f(m-1) = m-1$. This gives

$$f(2(m-1)) = 2f(m-1) = 2(m-1).$$

Since f is strictly increasing, using $f(m-1) = m-1$ and $f(2(m-1)) = 2(m-1)$, and noting that $m < 2(m-1)$, it follows that

$$f(m) = m.$$

By the principle of induction, the claim holds. □

■

Using $f(3) = 3$ and that

$$f(2^{n+1}+2) = 2f(2^n+1)$$

holds for any integer $n \geq 1$, one can also prove by induction that $f(2^n+1) = 2^n+1$ for any integer $n \geq 1$. The monotonicity of f then implies that $f(n) = n$ for all positive integers n.

Example 1.3 (India RMO 2006 P7)**.** Let X be the set of all positive integers greater than or equal to 8, and let $f\colon X \to X$ be a function such that $f(x+y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.

> **Walkthrough** — Using the given condition, try to express $f(9)$ in terms of $f(8)$.

Solution 3. Note that

$$f(9) = f(4+5) = f(20) = f(4+16) = f(64),$$

and

$$f(8) = f(4+4) = f(16) = f(8+8) = f(64)$$

hold. Using the given condition $f(8) = 9$, it follows that $f(9) = 9$. ■

Exercise 1.4 (INMO 2026 P4)**.** Two integers a and b are called *companions* if every prime number p either divides both or none of a, b. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(0) = 0$ and the numbers $f(m)+n$ and $f(n)+m$ are companions for all $m, n \in \mathbb{N}_0$. (Here $\mathbb{N}_0$ denotes the set of all non-negative integers.)

Walkthrough —
(a)

Solution 4. Let m, n be elements of $\mathbb{N}_0$. Let p be a prime number dividing $f(m)-f(n)$. Let k be a positive integer such that p divides the integers $f(m)+k$ and $f(n)+k$. Then p also divides the integers $m+f(k)$ and $n+f(k)$, and hence p divides $m-n$. Thus, for any prime number p, if p divides $f(m)-f(n)$, then p divides $m-n$.

In particular, for any integer $n \geq 0$, it follows that $f(n+1)-f(n)$ is equal to one of $1, -1$. We claim that $f(n+1)-f(n)=1$ for all $n \geq 0$. On the contrary, suppose that there exists an integer $n \geq 0$ such that $f(n+1)-f(n)=-1$. Let p be a prime number greater than or equal to $n+1$. This gives

$$\begin{aligned} f(p) &= \sum_{i=1}^{p} (f(i) - f(i-1)) \\ &\leq p-2. \end{aligned}$$

Since $f(p)+0, p+f(0)$ are companions, it follows that $f(p)$ is a positive multiple of p, which contradicts the inequality $f(p) \leq p-2$. Thus, we have $f(n+1)-f(n)=1$ for all $n \geq 0$, and hence $f(n)=n$ for all $n \geq 0$. It is easy to verify that this function satisfies the required conditions. Therefore, the only function satisfying the required conditions is

$$f(n) = n \quad \text{for all } n \in \mathbb{N}_0.$$

■

§1.1 Using symmetry

Example 1.5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4$$

holds for all $x \in \mathbb{R}$.

Solution 5. Let f be a function satisfying the given functional equation. Let x be an element of $\mathbb{R}$. Note that

$$\begin{aligned} x^2 f(x) + f(1-x) &= 2x - x^4, \\ f(x) + (1-x)^2 f(1-x) &= 2(1-x) - (1-x)^4 \end{aligned}$$

holds. Eliminating $f(1-x)$ from these two equations, we get

$$((x(1-x))^2 - 1) f(x) = (2x - x^4)(1-x)^2 - 2(1-x) + (1-x)^4$$

$$
\begin{aligned}
&= (1-x)\left((1-x)^3 + (2x-x^4)(1-x) - 2\right) \\
&= (1-x)\left(1 - 3x + 3x^2 - x^3 + 2x - 2x^2 - x^4 + x^5 - 2\right) \\
&= (1-x)\left(-1 - x + x^2 - x^3 - x^4 + x^5\right) \\
&= (1-x)(x+1)\left(-1 - x^3 + x^2(x^2 - x + 1)\right) \\
&= (1-x^2)(-1 - x^3 + x^4 - x^3 + x^2) \\
&= (1-x^2)(x^4 - 2x^3 + x^2 - 1) \\
&= (1-x^2)((x(1-x))^2 - 1).
\end{aligned}
$$

Note that

$$x(1-x) \leq \frac{(x+1-x)^2}{4} = \frac{1}{4}$$

holds. This yields

$$((x(1-x)) + 1)f(x) = (1-x^2)((x(1-x)) + 1).$$

If $x(1-x) + 1 \neq 0$, then it follows that

$$f(x) = 1 - x^2.$$

Claim — Let

$$\alpha = \frac{1+\sqrt{5}}{2},$$
$$\beta = \frac{1-\sqrt{5}}{2}$$

denote the roots of the polynomial $x^2 - x - 1$. Let λ denote a real number. Let $g_\lambda \colon \mathbb{R} \to \mathbb{R}$ be defined as

$$g_\lambda(x) = \begin{cases} 1 - x^2 & \text{if } x^2 - x - 1 \neq 0, \\ \lambda & \text{if } x = \alpha, \\ 2\alpha - \alpha^4 - \alpha^2\lambda & \text{if } x = \beta. \end{cases}$$

Then g_λ satisfies the given functional equation.

Proof of the Claim. Since the roots of the polynomial $x^2 - x - 1$ add up to 1, it follows that for a real number t, both t and $1 - t$ are roots of the polynomial or neither of them is. Thus, if an element $t \in \mathbb{R}$ is not a root of the polynomial $x^2 - x - 1$, then

$$
\begin{aligned}
t^2 g_\lambda(t) + g_\lambda(1-t) &= t^2(1-t^2) + (1 - (1-t)^2) \\
&= 2t - t^4.
\end{aligned}
$$

Then,

$$\begin{aligned}\alpha^2 g_\lambda(\alpha) + g_\lambda(1-\alpha) &= \alpha^2\lambda + g_\lambda(\beta)\\ &= \alpha^2\lambda + (2\alpha - \alpha^4 - \alpha^2\lambda)\\ &= 2\alpha - \alpha^4\end{aligned}$$

holds. Moreover, we have

$$\begin{aligned}\beta^2 g_\lambda(\beta) + g_\lambda(1-\beta) &= \beta^2(2\alpha - \alpha^4 - \alpha^2\lambda) + g_\lambda(\alpha)\\ &= \beta^2(2\alpha - \alpha^4 - \alpha^2\lambda) + \lambda\\ &= -2\beta - \alpha^2 \quad (\text{using } \alpha\beta = -1\ )\\ &= -2\beta - (1-\beta)^2\\ &= -2\beta - (1 - 2\beta + \beta^2)\\ &= -1 - \beta^2\\ &= 2\beta - (1+\beta)^2\\ &= 2\beta - \beta^4.\end{aligned}$$

This completes the proof of the Claim. □

Thus, the functions satisfying the given functional equation are precisely the functions g_λ for real numbers λ. ■

Example 1.6. Determine all functions $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ satisfying

$$f(x) + 2f\left(\frac{1}{x}\right) = x$$

for all nonzero real numbers x.

Solution 6. Let $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a function satisfying the given functional equation. Let x be a nonzero real number. Note that

$$\begin{aligned}f(x) + 2f\left(\frac{1}{x}\right) &= x,\\ 2f(x) + f\left(\frac{1}{x}\right) &= \frac{1}{x}\end{aligned}$$

holds. Eliminating $f(1/x)$ from these two equations, we get

$$-3f(x) = x - \frac{2}{x},$$

which yields

$$f(x) = \frac{2}{3x} - \frac{x}{3} = \frac{2-x^2}{3x}.$$

Moreover, the function $g\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ defined as

$$g(x)=\frac{2-x^2}{3x}$$

for any nonzero real number x satisfies the given functional equation. Indeed, for any nonzero real number x, we have

$$\begin{aligned} g(x)+2g\left(\frac{1}{x}\right) &= \frac{2-x^2}{3x}+2\cdot\frac{2-\frac{1}{x^2}}{3\cdot\frac{1}{x}} \\ &= \frac{2-x^2}{3x}+\frac{4x^2-2}{3x} \\ &= x. \end{aligned}$$

Thus, the only function satisfying the given functional equation is g. ■

Example 1.7. Determine all functions $f\colon \mathbb{R}\setminus\{0,1\}\to\mathbb{R}$ such that

$$f(x)+2f\left(\frac{1}{1-x}\right)+3f\left(1-\frac{1}{x}\right)=x$$

holds for all real numbers $x\neq 0,1$.

Solution 7. Let $f\colon \mathbb{R}\setminus\{0,1\}\to\mathbb{R}$ be a function satisfying the given functional equation. Let x be a real number such that $x\neq 0,1$. Observe that $\frac{1}{1-x}$ and $1-\frac{1}{x}$ are also real numbers different from 0 and 1. Note that

$$\begin{aligned} f(x)+2f\left(\frac{1}{1-x}\right)+3f\left(1-\frac{1}{x}\right) &= x, \\ 3f(x)+f\left(\frac{1}{1-x}\right)+2f\left(1-\frac{1}{x}\right) &= \frac{1}{1-x}, \\ 2f(x)+3f\left(\frac{1}{1-x}\right)+f\left(1-\frac{1}{x}\right) &= 1-\frac{1}{x} \end{aligned}$$

holds. This gives

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} f(x) \\ f\left(\frac{1}{1-x}\right) \\ f\left(1-\frac{1}{x}\right) \end{pmatrix}=\begin{pmatrix} x \\ \frac{1}{1-x} \\ 1-\frac{1}{x} \end{pmatrix}.$$

The inverse of the coefficient matrix is

$$\frac{1}{14}\begin{pmatrix} -5 & 4 & 3 \\ 3 & -5 & 4 \\ 4 & 3 & -5 \end{pmatrix}.$$

The content posted here and at this blog by Evan Chen are quite useful.

Thus, we have

$$\begin{pmatrix} f(x) \\ f\left(\frac{1}{1-x}\right) \\ f\left(1-\frac{1}{x}\right) \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{1-x} \\ 1-\frac{1}{x} \end{pmatrix}.$$

In particular, we have

$$\begin{aligned} f(x) &= \frac{1}{18}\left(-5x + \frac{7}{1-x} + 1 - \frac{1}{x}\right) \\ &= \frac{-5x^2(1-x) + 7x - (x-1)^2}{18x(1-x)} \\ &= \frac{5x^3 - 5x^2 + 7x - x^2 + 2x - 1}{18x(1-x)} \\ &= \frac{5x^3 - 6x^2 + 9x - 1}{18x(1-x)}. \end{aligned}$$

Moreover, the function $g\colon \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$ defined as

$$g(x) = \frac{5x^3 - 6x^2 + 9x - 1}{18x(1-x)}$$

for any real number $x \neq 0, 1$ satisfies the given functional equation. Indeed, for any real number $x \neq 0, 1$, the real numbers $\frac{1}{1-x}$ and $1 - \frac{1}{x}$ are different from 0 and 1, and we have

$$\begin{aligned} &g(x) + 2g\left(\frac{1}{1-x}\right) + 3g\left(1-\frac{1}{x}\right) \\ &= \frac{5x^3 - 6x^2 + 9x - 1}{18x(1-x)} + 2 \cdot \frac{5 - 6(1-x) + 9(1-x)^2 - (1-x)^3}{-18x(1-x)} \\ &\quad + 3 \cdot \frac{5(x-1)^3 - 6x(x-1)^2 + 9x^2(x-1) - x^3}{-18x(1-x)} \\ &= \frac{1}{18x(1-x)}\big(5x^3 - 6x^2 + 9x - 1 \\ &\quad - 2\left(5 - 6 + 9 - 1 + 6x - 18x + 3x + 9x^2 - 3x^2 + x^3\right) \\ &\quad - 3\left(5x^3 - 6x^3 + 9x^3 - x^3 - 15x^2 + 12x^2 - 9x^2 + 15x - 6x - 5\right)\big) \\ &= \frac{1}{18x(1-x)}\big(5x^3 - 6x^2 + 9x - 1 - 2\left(x^3 + 6x^2 - 9x + 7\right) \\ &\quad - 3\left(7x^3 - 12x^2 + 9x - 5\right)\big) \\ &= \frac{18x^2 - 18x^3}{18x(1-x)} \\ &= x. \end{aligned}$$

This shows that the only function satisfying the given functional equation is g. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Tao06**] Terence Tao. *Solving mathematical problems*. A personal perspective. Oxford University Press, Oxford, 2006, pp. xii+103. ISBN: 978-0-19-920560-8; 0-19-920560-4 (cited p. 2)