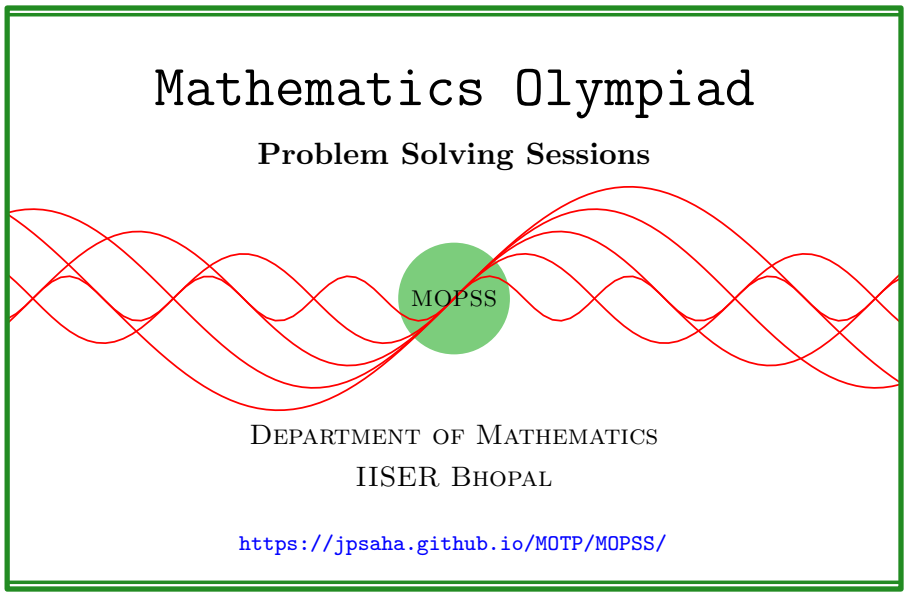
# Binomial theorem

MOPSS




## Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

# List of problems and examples


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# §1 Binomial theorem

**Example 1.1** (India RMO 1996 P6)**.** Given any positive integer $n$, show that there are two positive rational numbers $a$ and $b$, $a \neq b$, which are not integers and which are such that $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers.

**Walkthrough** —

**(a)** We may expect that $a = c + \frac{1}{2}$ and $b = d + \frac{1}{2}$ would work where $c, d$ are suitable integers. We need to see for what choices of $c, d$, the difference $a^m - b^m$ is an integer for all $1 \leq m \leq n$.

**(b)** By the binomial theorem [a],

$$a^m = c^m + \binom{m}{1} c^{m-1}\frac{1}{2} + \binom{m}{2} c^{m-2}\frac{1}{2^2} + \cdots + \binom{m}{m-1} c\frac{1}{2^{m-1}} + \frac{1}{2^m},$$

$$b^m = d^m + \binom{m}{1} d^{m-1}\frac{1}{2} + \binom{m}{2} d^{m-2}\frac{1}{2^2} + \cdots + \binom{m}{m-1} d\frac{1}{2^{m-1}} + \frac{1}{2^m}.$$

This shows

$$a^m - b^m = (c^m - d^m) + \binom{m}{1}\frac{c^{m-1}-d^{m-1}}{2} + \binom{m}{2}\frac{c^{m-2}-d^{m-2}}{2^2} + \cdots + \binom{m}{m-1}\frac{c-d}{2^{m-1}}.$$

**(c)** It follows that $a^m - b^m$ would be an integer if we could ensure that

$$\frac{c^{m-1}-d^{m-1}}{2}, \frac{c^{m-2}-d^{m-2}}{2^2}, \ldots, \frac{c-d}{2^{m-1}}$$

are all integers.

**(d)** Note that $c^{m-1}-d^{m-1}, c^{m-2}-d^{m-2}, \ldots, c-d$ are all divisible by $c-d$. So it would be enough to find suitable integers $c, d$ such that $c-d$ is divisible by $2, 2^2, \ldots, 2^{m-1}$ for all $1 \leq m \leq n$.

**(e)** For instance, if we take $c = 2^{n-1}, d = 0$, that is,

$$a = 2^{n-1} + \frac{1}{2}, b = \frac{1}{2},$$

then $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers and $a, b$ are positive rationals and not integers.

[a]The binomial theorem says that

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n,$$

where $\binom{n}{i}$ is a positive integer and is equal to $\frac{n!}{i!(n-i)!}$ for $1 \le i \le n-1$. Indeed, if we expand $(x+y)^n$, then it becomes immediately clear that there exist integers $c_1, \ldots, c_{n-1}$ such that

$$(x+y)^n = x^n + c_1x^{n-1}y + c_2x^{n-2}y^2 + \cdots + c_{n-2}x^2y^{n-2} + c_{n-1}xy^{n-1} + y^n$$

holds, and moreover, the integers $c_1, \ldots, c_{n-1}$ do not depend on $x$ and $y$, they depend only on $n$. A careful inspection of the above equality shows that for any $1 \le i \le n$, the integer $c_i$ is equal to the number of ways of selecting a set of $i$ elements from a set of $n$ elements, which is denoted by $\binom{n}{i}$ by convention. This essentially proves (combined with the details to be filled in) the binomial theorem, which states that for any positive integer $n$,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n$$

holds. However, it is not immediate that $\binom{n}{i}$ is equal to $\frac{n!}{i!(n-i)!}$. But we do not require it in this problem.

**Solution 1.** Take $a = 2^m + \frac{1}{2}$ and $b = \frac{1}{2}$, where $m$ is a suitable positive integer to be determined later. Using the binomial theorem, we obtain

$$a^k = 2^{mk} + \binom{k}{1}\frac{2^{m(k-1)}}{2} + \binom{k}{2}\frac{2^{m(k-2)}}{2^2} + \cdots + \binom{k}{k-1}\frac{2^m}{2^{k-1}} + \frac{1}{2^k}$$

for any integer $k \ge 1$. We would like to have that $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers. Note that it suffices to make sure that $m \ge k-1$ for $k = 1, 2, \ldots, n$. Let us take

$$a = 2^n + \frac{1}{2}, b = \frac{1}{2}.$$

Note that the positive rational numbers $a, b$ are not integers, and $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers. ■

**Exercise 1.2** (Putnam 2004 B2, AoPS)**.** Let $m$ and $n$ be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^mn^n}.$$

**Walkthrough** —

**(a)** Apply the binomial theorem.

**Solution 2.** Applying the binomial theorem, we observe that $\binom{m+n}{m}m^mn^n$ is one of the terms of the binomial expansion of $(m+n)^{m+n}$. Noting that $\binom{m+n}{m}m^mn^n = \frac{(m+n)!}{m!n!}m^mn^n$, the result follows. ■

## References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)